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Berry phase for spin-1/2 particles moving in a space-time with torsion

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Abstract. Berry phase for a spin-1/2 particle moving in a flat space-time with torsion is investigated in the context of the Einstein–Cartan–Dirac model. It is shown that if the torsion is due to a dense polarized background, then there is a Berry phase only if the fermion is massless and its momentum is perpendicular to the direction of the background polarization. The order of magnitude of this Berry phase is discussed in other theoretical frameworks.

1 Introduction

The geometry of a four-dimensional space-time U^4 is given by the metric ($g_{\mu\nu}$) and torsion ($T^{\alpha}_{\mu\nu}$) tensors. In the context of Einstein–Cartan–Dirac (ECD) theory, the axial current of the background material is the source of the torsion field and leads to a completely anti-symmetric torsion tensor, represented by a pseudo-vector S^{μ}. This pseudo-vector is coupled to the axial current of all the fermion species. (For a brief review of Einstein–Cartan–Dirac theory, see Appendix A.)

One of the most important features of the physics of torsion is its phenomenological aspects. To study this, we must realize that in the context of ECD theory, the torsion of space-time vanishes in vacuum, and one expects a non-vanishing torsion pseudo-vector only if the space is filled with a (spin-) polarized background matter. But in such a space-time, there are plenty of interactions which can easily mask any effect of torsion. So the best candidate to probe such an interaction is the neutrino, as it is weakly coupled to the rest of the matter.

Studies of the interactions of neutrinos with torsion go to several years ago. In [1] (see also [2]), the effect of torsion on neutrino oscillations has been studied by assuming that the torsion eigenstates, i.e. the eigenstates of the interaction part of the Hamiltonian, are different from the weak interaction eigenstates. Hammond has studied the different aspects of the fermion interaction with a torsion field derived from a second rank potential [3,4], for example the torsion coupling constant [5] and the relation between the intrinsic spin of the string and the torsion [6]. The theoretical and phenomenological aspects of the torsion field have been investigated in [7] by an effective field theoretical method, and the contribution of the torsion of space-time to standard neutrino oscillations has been studied in the context of ECD theory in [8], in which the torsion and weak interaction eigenstates have been considered the same. More recently, the quantum reflection of a massless neutrino from a torsion induced potential barrier has been discussed in [9].

Before going further, it may be useful to describe why we consider the ECD theory to investigate the physics of torsion and why we do not work in a more general framework in which the torsion field is considered as an propagating quantum field. The reason, in our view, is that if one considers the torsion field S^{μ} as a quantum field which propagates, the resulting theory will have serious problems. As has been shown in [10], the effective quantum field theory of a massive fermion coupled to the axial vector S^{μ} (i.e. the torsion field) is unitary and renormalizable only when $m \ll M$; that is, when the torsion mass is much greater than the mass of the heaviest fermion. But the restrictions coming from the contact experiment reach only the region $M < 3\,\text{TeV}$ [7], which is not enough to satisfy the condition $m \ll M$ for all the fermions of the standard model. Therefore the ECD theory is almost the unique available quantum theory of gravity with torsion.

In general, the evolution of the neutrinos in a space-time is affected by

(1) the structure of the mass matrix (normally leading to oscillations);
(2) effects of matter (weak interaction, MSW effect, etc.);
(3) gravity (i.e. metric); and
(4) torsion of space-time.

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In general, both the amplitude and the phase of the neutrino wavefunctions can be affected by all these effects. Now if there is some nontrivial contribution to the phase of the neutrino wavefunction due to the torsion field – the so-called Berry phase – then there might be some detectable effect associated with torsion. So it is worthwhile to investigate the Berry phase of a spin-1/2 particle in a space-time with torsion.

In this paper, we want to study this Berry phase in the context of ECD theory. We consider a space-time U^4, whose metric is $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$, and its torsion pseudo-vector S^μ is due to static polarized dense matter[1]. We see that $S^\mu = (0, K'\hat{\mathbf{s}})$, where K' is some constant depending on the specific model considered – through the coupling constant – and also on the density and polarization of the background matter. The unit vector $\hat{\mathbf{s}}$ determines the direction of the background polarization (spin).

2 Dirac equation in U^4

The Hamiltonian for a spin-1/2 particle moving in this U^4 space-time is $H = H_0 + H_1$ (see (A.9)), where $H_0 = c\boldsymbol{\alpha} \cdot \boldsymbol{P} + mc^2\beta$ is the usual Hamiltonian in a flat (Minkowski) space-time (α_1, α_2, α_3, and β are the Dirac matrices), and $H_1 = (\hbar c/8)\gamma_5\gamma^0 \not{S}$ is the interaction Hamiltonian due to the torsion field.

It is important to note that both H_0 and H_1 are Hermitian. Therefore, the time evolution generated by the total Hamiltonian is unitary. So the conclusion of [11, 12] where the authors have obtained a "dissipative term", which causes the state to decrease exponentially, is wrong. Their mistake, we think, is in calculating H_1 ((10) of [12]). More recently, the effect of the Berry phase on neutrino oscillations has been studied in [13]. In these papers, the authors have considered a null vector S^μ, which is different from ours (which is derived in the framework of ECD theory). Also they have considered massive neutrinos, which again is different from the situation we study. As we will show, we have to consider massless neutrinos (which cannot oscillate).

In order to study the Berry phase of a spin-1/2 particle, we must first calculate the torsion field S^μ. For simplicity, we consider a fermionic medium with all the fermions *at rest*, through which our spin-1/2 particle moves. In this case, it can be shown that the torsion field S^μ is $S^\mu = (0, K'\hat{\mathbf{s}})$, where K' in the context of ECD theory is $-48\pi\hbar\rho G/c^3$, and ρ and $\hat{\mathbf{s}}$ are the number density and polarization unit vector of the background matter, respectively [9]. Now choosing the chiral representation for the Dirac matrices, these two Hamiltonians read

$$H_0 = \begin{pmatrix} -c\boldsymbol{\sigma} \cdot \boldsymbol{P} & mc^2 \\ mc^2 & c\boldsymbol{\sigma} \cdot \boldsymbol{P} \end{pmatrix},$$

$$H_1 = K \begin{pmatrix} \boldsymbol{\sigma} \cdot \hat{\mathbf{s}} & 0 \\ 0 & \boldsymbol{\sigma} \cdot \hat{\mathbf{s}} \end{pmatrix}, \tag{1}$$

where the coupling constant K is (in the ECD model)

$$K_{\mathrm{ECD}} = \frac{12\pi\rho G\hbar^2}{c^2}. \tag{2}$$

Here $\boldsymbol{P}$ is the momentum of the spin-1/2 particle, and the σ_i are the Pauli matrices. Let us take $\boldsymbol{P} = P\hat{\mathbf{z}}$ and $\hat{\mathbf{s}} = \sin\varphi\cos\alpha\hat{\mathbf{x}} + \sin\varphi\sin\alpha\hat{\mathbf{y}} + \cos\varphi\hat{\mathbf{z}}$. Therefore the total Hamiltonian becomes

$$H = \begin{pmatrix} -cP + Ks_3 & Ks_- & mc^2 & 0 \\ Ks_+ & cP - KS_3 & 0 & mc^2 \\ mc^2 & 0 & cP + Ks_3 & Ks_- \\ 0 & mc^2 & Ks_+ & -cP - Ks_3 \end{pmatrix}, \tag{3}$$

where $s_\pm := s_1 \pm \mathrm{i}s_2$ and $s_3 := s_z$.

3 Evolution of the particle state and its Berry phase

We consider the following problem: An eigenstate of H_0 begins to move in U^4. This can be the case if, for example, there is a region of space where the matter is polarized and a spin-1/2 particle enters this region; or if in such a region, a particle is created as an eigenstate of H_0.

With no loss of generality, we can choose $\hat{\mathbf{s}}$ in the xz plane. In this case, the eigenvalues of H are E_1, $-E_1$, E_2, and $-E_2$, where

$$E_1 = \sqrt{K^2 + c^2P^2 + m^2c^4 + 2cK\sqrt{m^2c^2 + P^2\cos^2\varphi}}, \tag{4}$$

$$E_2 = \sqrt{K^2 + c^2P^2 + m^2c^4 - 2cK\sqrt{m^2c^2 + P^2\cos^2\varphi}}. \tag{5}$$

Let $|\psi(0)\rangle$ be the following eigenstate of H_0:

$$|\psi(0)\rangle = \begin{pmatrix} 0 & 1 & 0 & (\sqrt{q^2 + m^2c^2} - q)/mc \end{pmatrix}^{\mathrm{t}}, \tag{6}$$

where q is the momentum of the particle in the torsion-free region and "t" denotes the transpose. For $m = 0$, this state becomes $|\psi(0)\rangle = \begin{pmatrix} 0 & 1 & 0 & 0 \end{pmatrix}^{\mathrm{t}}$, which is the spinor of a left-handed neutrino. $|\psi(0)\rangle$ in (6) can be written as a superposition of $|E_1\rangle$, $|-E_1\rangle$, $|E_2\rangle$, and $|-E_2\rangle$:

$$|\psi(0)\rangle = a|E_1\rangle + b|-E_1\rangle + c|E_2\rangle + d|-E_2\rangle. \tag{7}$$

At time t, this state becomes

$$|\psi(t)\rangle = a\mathrm{e}^{-\mathrm{i}E_1t/\hbar}|E_1\rangle + b\mathrm{e}^{\mathrm{i}E_1t/\hbar}|-E_1\rangle + c\mathrm{e}^{-\mathrm{i}E_2t/\hbar}|E_2\rangle + d\mathrm{e}^{\mathrm{i}E_2t/\hbar}|-E_2\rangle. \tag{8}$$

From the general theory of the Berry phase [14], we know that if at some time t, say T, $|\psi(T)\rangle = \mathrm{e}^{\mathrm{i}\Phi}|\psi(0)\rangle$, then there will be a Berry phase $\beta = \Phi + \mathrm{i}\int_0^T \mathrm{d}t\langle\psi(t)|\mathrm{d}/\mathrm{d}t|\psi(t)\rangle$. In our problem, such a T exists only if $\exp(2\mathrm{i}E_1T/\hbar) =$

[1] Actually, for torsion to be important, the matter must be very dense; and in such a situation the metric shall not be flat. We are considering a flat metric to see the effect of a pure torsion. Any calculation must be repeated in the more general case of a curved metric, such as the Schwarzschild metric

$\exp(\mathrm{i}(E_1-E_2)T/\hbar) = \exp(\mathrm{i}(E_1+E_2)T/\hbar) = 1$, from which it follows that $E_2 = (2n+1)E_1$, for some integer n. In other words,

$$4n(n+1)(K^2 + c^2P^2 + m^2c^4) + 4(2n^2 + 2n + 1)cK\sqrt{m^2c^2 + P^2\cos^2\varphi} = 0. \tag{9}$$

This is independent of P only if $n = 0$. We then get $K(m^2c^2 + P^2\cos^2\varphi)^{1/2} = 0$, which is true in either of the following two cases:

(1) $K = 0$, i.e., when there is no torsion. This is the case of a free particle moving in ordinary, i.e. torsion-free, Minkowski space-time. In this case there is no Berry phase, the phase is only dynamical.

(2) $m^2c^2 + P^2\cos^2\varphi = 0$, which is true only if $m = 0$ and $\varphi = \pi/2$. In the following, we calculate the Berry phase for this nontrivial case, i.e. a massless fermion with momentum perpendicular to the polarization of the background.

To calculate the Berry phase in case (2) above, we need the eigenstates $|E_1\rangle$, $|-E_1\rangle$, $|E_2\rangle$, and $|-E_2\rangle$ (when $m = 0$ and $\varphi = \pi/2$, we have $E_1 = E_2 = E$)

$$|E_1\rangle = \begin{pmatrix} \dfrac{-cP+E}{\sqrt{K^2+(cP-E)^2}} \\ \dfrac{K}{\sqrt{K^2+(cP-E)^2}} \\ 0 \\ 0 \end{pmatrix}, \quad |-E_1\rangle = \begin{pmatrix} 0 \\ 0 \\ \dfrac{cP-E}{\sqrt{K^2+(cP-E)^2}} \\ \dfrac{K}{\sqrt{K^2+(cP-E)^2}} \end{pmatrix}, \tag{10}$$

$$|E_2\rangle = \begin{pmatrix} 0 \\ 0 \\ \dfrac{cP+E}{\sqrt{K^2+(cP+E)^2}} \\ \dfrac{K}{\sqrt{K^2+(cP+E)^2}} \end{pmatrix}, \quad |-E_2\rangle = \begin{pmatrix} -\dfrac{cP+E}{\sqrt{K^2+(cP+E)^2}} \\ \dfrac{K}{\sqrt{K^2+(cP+E)^2}} \\ 0 \\ 0 \end{pmatrix}, \tag{11}$$

where $E := (c^2P^2 + K^2)^{1/2}$ is the energy of $|\psi(0)\rangle$.

Expanding (6) (with $m = 0$) in terms of these spinors, and using the notation of (7), we see that $b = c = 0$, and

$$\mathrm{i}\langle\psi(t)|\mathrm{d}/\mathrm{d}t|\psi(t)\rangle = \frac{E}{\hbar}\left(|a|^2 - |d|^2\right) = \frac{1}{\hbar}\sqrt{E^2 - K^2}. \tag{12}$$

Also in this case, Φ and T are

$$\Phi = -\pi, \quad T = \frac{\pi\hbar}{E}. \tag{13}$$

Therefore, the Berry phase β becomes

$$\beta = \pi\left(\sqrt{1-(K/E)^2} - 1\right) = -\frac{\pi}{2}\frac{K^2}{E^2} + \cdots \tag{14}$$

This means that $\lim_{K\to 0}\beta = 0$, as we expect. Note that K has the dimension of energy.

The order of magnitude of this effect depends on the coupling constant and the total axial current of the background matter. In Einstein–Cartan–Dirac theory (minimal coupling), K is given by (2), which leads to the small value K_{ECD} (eV) $\sim 10^{-69}\rho$ (cm^{-3}). But in other theoretical frameworks, it may lead to greater values. For example in some models, the Newton gravitational constant is replaced by the weak coupling constant $G_T \sim 10^{31}G$ (see for example [2]), which leads to $K_{\mathrm{V-A}}$ (eV) $\sim 10^{-38}\rho$ (cm^{-3}). In the effective field theory approach of Belyaev and Shapiro [7], the value of K could be as large as K_{EFT} (eV) $\sim 10^{-38}\rho$ (cm^{-3}). And finally in the strong gravity regime (i.e. inside collapsing matter or in the early stage of the universe), $G \to G_{\mathrm{SG}} \sim 10^{39}G$ [15], so K_{SG} (eV) $\sim 10^{-30}\rho$ (cm^{-3}). For a more detailed discussion of the torsion coupling constant, see [9] and references therein.

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Appendix

A Brief review of Einstein–Cartan–Dirac theory

The geometry of a d-dimensional space-time U^d is given by two geometrical objects: a metric tensor $g_{\mu\nu}$ and a connection $\Gamma^{\alpha}_{\mu\nu}$. The most general metric-compatible connection is $\Gamma^{\alpha}_{\mu\nu} = \left\{{\alpha \atop \mu\nu}\right\} + K^{\alpha}_{\mu\nu}$, where $\left\{{\alpha \atop \mu\nu}\right\}$ is the usual Christoffel symbol, derived from the metric, and $K^{\alpha}_{\mu\nu}$ is a tensor of rank 3, named contorsion. $K^{\alpha}_{\mu\nu}$ is related to the torsion tensor $T^{\alpha}_{\mu\nu} := \Gamma^{\alpha}_{\mu\nu} - \Gamma^{\alpha}_{\nu\mu}$ as follows:

$$K_{\alpha\mu\nu} = \frac{1}{2}(T_{\alpha\mu\nu} - T_{\mu\alpha\nu} - T_{\nu\alpha\mu}). \tag{A.1}$$

$T_{\alpha\mu\nu}$ can be decomposed as

$$T_{\alpha\mu\nu} = \frac{1}{3}\left(g_{\alpha\nu}T_{\mu} - g_{\alpha\mu}T_{\nu}\right) - \frac{1}{6}\varepsilon_{\alpha\mu\nu\sigma}S^{\sigma} + q_{\alpha\mu\nu}, \tag{A.2}$$

where $T_{\mu} := -g^{\alpha\beta}K_{\alpha\beta\mu}$, $S^{\sigma} := -\varepsilon^{\sigma\alpha\mu\nu}K_{\alpha\mu\nu}$, and $q_{\alpha\mu\nu}$ is the remainder, defined by (A.2). Using the usual procedure, one can show that the scalar curvature R of this space-time is

$$R = \tilde{R} - \frac{2}{\sqrt{-g}}\partial_{\kappa}\left(\sqrt{-g}\tau^{\kappa}\right) - \left(\frac{4}{3}T^2 + \frac{1}{24}S^2 + \frac{1}{2}q_{\alpha\mu\nu}q^{\mu\nu\alpha}\right), \tag{A.3}$$

where $\tilde{R}$ is the Ricci scalar derived from the Christoffel symbols, i.e. the scalar curvature of the torsion-free space-time.

In Einstein–Cartan–Dirac theory, the space-time is assumed to be U^4, with both metric and torsion. The fields of the theory are the metric, the contorsion, and a set of spin-1/2 fields which are minimally coupled to the metric and torsion by the usual covariant derivative. The total action of Einstein–Cartan–Dirac theory is $I = I_{\rm EC} + I_{\rm D}$, where

$$I_{\rm EC} = -\frac{c^3}{16\pi G}\int {\rm d}^4x\sqrt{-g}R, \tag{A.4}$$

and

$$I_{\rm D} = {\rm i}\hbar\sum_j\int {\rm d}^4x\sqrt{-g}\bar{\psi}_j\left(e^\mu_a\gamma^a(\partial_\mu + \Gamma_\mu) + {\rm i}\frac{m_jc}{\hbar}\right)\psi_j, \tag{A.5}$$

where the sum is over all fermions species. Variation with respect to the contorsion field leads to

$$S^\mu = \frac{72\pi\hbar G}{c^3}\sum_j(J_j)^\mu_5, \tag{A.6}$$

$$T^\mu = 0, \quad q_{\alpha\mu\nu} = 0. \tag{A.7}$$

This means that, the contorsion (or torsion) field is completely anti-symmetric, and the pseudo-vector dual to the torsion field is the sum of the axial currents of the fermion field(s) ($(J_j)^\mu_5 = \bar{\psi}_j\gamma^\mu\gamma_5\psi_j$).

Variation with respect to the fermion fields leads to the following Dirac equation:

$$\gamma^\mu\partial_\mu\psi_j + {\rm i}\frac{m_jc}{\hbar}\psi_j + \frac{\rm i}{8}\gamma_5 \not{S}\psi_j = 0. \tag{A.8}$$

This equation can be written as a Schrödinger-type equation: ${\rm i}\hbar\partial_t\psi = H\psi$, where for $g_{\mu\nu} = \eta_{\mu\nu}$ the Hamiltonian H becomes

$$H = c\boldsymbol{\alpha}\cdot\boldsymbol{P} + mc^2\beta + \frac{\hbar c}{8}\gamma_5\gamma^0\not{S}. \tag{A.9}$$

In the above equations, S^μ is the torsion pseudo-vector which in the Einstein–Cartan–Dirac theory is given by (A.6).

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